Gauss Test for Convergence

3. D' Alembert's Ratio Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(iii) Test fails if $l = 1$

4. Raabe's Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

(iii) Test fails if $l = 1$

Gauss Test :- $\sum u_n$ be a series of +ve terms s.t

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p}$$

$\alpha > 0$ $\quad p > 1$ $\quad \langle r_n \rangle$ Bounded.

If $\alpha \neq 1$

$\alpha > 1$ $\quad \sum u_n$ Cgt

$\alpha < 1$ $\quad \sum u_n$ divergent

If $\alpha = 1$.

$\beta > 1$ $\quad \sum u_n$ Cgt

$\beta \leq 1$ $\quad \sum u_n$ Divergent.

Exp

$$u_n = \frac{2^2 \cdot 4^2 \text{ ------ } (2n-2)^2}{1^2 \cdot 3^2 \text{ ------ } (2n-1)^2}$$

$$u_{n+1} = \frac{2^2 \cdot 4^2 \text{ ------ } (2n-2)^2 (2n)^2}{1^2 \cdot 3^2 \text{ ------ } (2n-1)^2 (2n+1)^2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^2 \cdot 4^2 \text{ ---- } (2n-2)^2 (2n)^2}{1^2 \cdot 3^2 \text{ ---- } (2n-1)^2 (2n+1)^2} \times \frac{1^2 \cdot 3^2 \text{ ---- } (2n-1)^2}{2^2 \cdot 4^2 \text{ ---- } (2n-2)^2}$$

$$\lim_{n\to\infty} \frac{(2n)^2}{(2n+1)^2} = \lim_{n\to\infty} \frac{\cancel{(2n)^2}}{\cancel{(2n)^2}\left[1+\frac{1}{2n}\right]^2} = 1.$$

⇒ Ratio test fails to find the Convergence

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{2^2\cdot 4^2 \cdots (2n-2)^2}{1^2\cdot 3^2 \cdots (2n-1)^2} \times \frac{1^2\cdot 3^2 \cdots (2n-1)^2(2n+1)^2}{2^2\cdot 4^2 \cdots (2n-2)^2(2n)^2} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n+1)^2}{(2n)^2} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{4n^2} + 1 + 4n - \cancel{4n^2}}{4n^2}\right]$$

$$= \lim_{n\to\infty} \cancel{n}\cdot\cancel{n}\left[\frac{4 + 1/n}{4\cancel{n^2}}\right] = 1.$$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = 1.$$

Rabbe's Test fail.

Gauss Test

$$\frac{u_n}{u_{n+1}} = \frac{2^2 \cdot 4^2 \text{---} (2n-2)^2}{1^2 \cdot 3^2 \text{---} (2n-1)^2} \times \frac{1^2 \cdot 3^2 \text{---} (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 \text{---} (2n-2)^2 (2n)^2}$$

$$= \frac{(2n+1)^2}{(2n)^2} = \frac{4n^2 + 1 + 4n}{4n^2}$$

$$= \frac{4\cancel{n^2}}{4\cancel{n^2}} + \frac{1}{4n^2} + \frac{\cancel{4}\cancel{n}}{\cancel{4}n^{\cancel{2}}}$$

$$= 1 + \frac{1}{n} + \frac{1}{4n^2}$$

Compare with

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p}$$

$\alpha = 1 > 0$

$p > 1$

$\underline{p = 2 > 1}$

$\langle \gamma_n \rangle$ Bounded

$\gamma_n = \frac{1}{4}$ Bounded.

Now $\alpha = 1.$

$\beta = 1.$

$\Rightarrow$ By Gauss Test

Σu_n is Divergent.