Limit and Continuity

Example

Let $f(x) = \frac{|x|}{x}$ for all except 0

Prove that $\lim_{x \to 0} f(x)$ does not exist.

Sol. Let $\epsilon > 0$ where $0 < \epsilon < 1$

Let $\delta > 0$ be any real number.

$\therefore$ $x \to 0$

$\therefore$ We take $0 < |x_1 - 0| < \delta$

$$0 < |x_2 - 0| < \delta \quad \text{for} \quad \begin{bmatrix} x_1 > 0 \\ x_2 < 0 \end{bmatrix}$$

$$|f(x_1) - f(x_2)| = \left| \frac{|x_1|}{x_1} - \frac{|x_2|}{x_2} \right|$$

$$= \left| \frac{x_1}{x_1} - \frac{(-x_2)}{x_2} \right| \quad \begin{bmatrix} |x_1| = x_1 & x_1 > 0 \\ |x_2| = -x_2 & x_2 < 0 \end{bmatrix}$$

$$= \left| \frac{x_1}{x_1} + \frac{x_2}{x_2} \right| = |1+1| = 2 > 1 > \epsilon$$

$$\Rightarrow |f(x_1) - f(x_2)| > \epsilon \quad \text{for} \quad \begin{array}{l} 0 < |x_1 - 0| < \delta \\ 0 < |x_2 - 0| < \delta \quad [0 < \epsilon < 1] \end{array}$$

$\therefore$ By Cauchy Criterian.

$\lim_{x \to 0} f(x)$ does not exist.

(ii) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R}-\mathbb{Q} \end{cases}$

Prove By using def.

$\lim_{x \to c} f(x)$ does not exist.