Limit and Continuity

Example

If $f(x) = x\left[\frac{1}{x}\right]$, then show that $\lim\limits_{x\to 1/3} f(x)$ does not exist.

Sol. Let $\frac{1}{x} = y$.

$$\lim_{x\to 1/3^-} f(x) = \lim_{y\to 3^+} \frac{1}{y}[y]$$

$x \to \frac{1}{3}^-$

3

$x < \frac{1}{3}$

$1/3$

$\frac{1}{x} > 3$

$y > 3,\ y \to 3^+$

$$= \lim_{y \to 3^+} \frac{1}{y} \cdot \lim_{y \to 3^+} [y]$$

$$= \lim_{\delta \to 0} \frac{1}{3+\delta} \cdot 3 = \frac{1}{3} \cdot 3 = 1.$$

$$\left[\because \lim_{x \to a^+} f(x) = \lim_{\delta \to 0} f(a+\delta) \quad \lim_{x \to a^+} [x] = a. \right]$$

$$\lim_{x \to 1/3^-} f(x) = 1. \quad \text{--- (1)}$$

$$\lim_{x \to \frac{1}{3}^+} f(x)$$

$x \to 1/3^+$

$x > \frac{1}{3}$

$\frac{1}{x} < 3$

$y < 3.$

$y \to 3^-$

$$= \lim_{y \to 3^-} \left(\frac{1}{y}\right)[y]$$

$$= \lim_{y \to 3^-} \frac{1}{y} \cdot \lim_{y \to 3^-} [y]$$

$$= \lim_{\delta \to 0} \frac{1}{3-\delta} (3-1)$$

$$= \frac{1}{3}(2) = \frac{2}{3}.$$

$$\lim_{x \to a^-} f(x) = \lim_{\delta \to 0} f(a-\delta)$$

$$\lim_{x \to a^-} [x] = a-1.$$

$$\lim_{x \to \frac{1}{3}^+} f(x) = \frac{2}{3} \quad \text{— (II)}$$

from (I) and (II)

$$\lim_{x \to 1/3^-} f(x) \neq \lim_{x \to 1/3^+} f(x)$$

$\Rightarrow \lim_{x \to 1/3} f(x)$ does not exist.

Hence Proved.