THE RIEMANN-STIELTJES INTEGRAL

$f_1, f_2 \in R(\alpha)$ on $[a,b]$ then

$(f_1 + f_2) \in R(\alpha)$ on $[a,b]$ also

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha.$$

<u>Proof</u> $\quad f_1, f_2 \in R(\alpha)$

$\therefore \exists$ partitions P_1 and P_2 s.t.

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \epsilon/2$$

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \epsilon/2$$

Let $P = P_1 \cup P_2$ is Common Refinement of P_1 and P_2

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon/2$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon/2$$

Let m_i and M_i are Bounds of $f(x) = f_1(x) + f_2(x)$

also m'_i and M'_i are Bounds of $f_1(x)$

m''_i and M''_i are Bounds of $f_2(x)$

$m_i \geq m'_i + m''_i$ — (i) and $M_i \leq M'_i + M''_i$ — (ii) from (1)

$$\sum_{i=1}^{n} m_i \Delta\alpha_i \geq \sum_{i=1}^{n} m'_i \Delta\alpha_i + \sum_{i=1}^{n} m''_i \Delta\alpha_i$$

$$L(P, f, \alpha) \geq L(P, f_1, \alpha) + L(P, f_2, \alpha)$$

from (1) $-L(P, f, \alpha) \leq -L(P, f_1, \alpha) - L(P, f_2, \alpha)$ — (IV)

$$\sum_{i=1}^{n} M_i \Delta\alpha_i \leq \sum_{i=1}^{n} M'_i \Delta\alpha_i + \sum_{i=1}^{n} M''_i \Delta\alpha_i$$

$$U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha) \quad \text{— (III)}$$

Add (II) and (IV)

$$U(P,f,\alpha) - L(P,f,\alpha) \leq U(P,f_1,\alpha) + U(P,f_2,\alpha) - L(P,f_1,\alpha) - L(P,f_2,\alpha)$$

$$= [U(P,f_1,\alpha) - L(P,f_1,\alpha)] + [U(P,f_2,\alpha) - L(P,f_2,\alpha)]$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$$

$$\Rightarrow f \in R(\alpha)$$

$$\Rightarrow (f_1 + f_2) \in R(\alpha)$$

$$\int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha \le U(P, f, \alpha) \quad \left[\overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)\right]$$

$$\le U(P_1, f_1, \alpha) + U(P_2, f_2, \alpha)$$

$$< L(P_1, f_1, \alpha) + \epsilon/2 + L(P_2, f_2, \alpha) + \epsilon/2$$

$$= L(P_1, f_1, \alpha) + L(P_2, f_2, \alpha) + \epsilon.$$

$$\le \underline{\int_a^b} f_1\,d\alpha + \underline{\int_a^b} f_2\,d\alpha + \epsilon \quad \left[\because \underline{\int_a^b} f\,d\alpha = \sup\{L(P,f,\alpha)\}\right]$$

$$= \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha + \epsilon \quad [f \in R(\alpha)]$$

where ϵ is arbitrary a small value.

$$\therefore \int_a^b f\,d\alpha \leq \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha \quad \text{—(i)}$$

Now take f_1 and f_2 are $-f_1$ and $-f_2$ respectively.

$$\int_a^b f\,d\alpha \geq \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha. \quad \text{—(ii)}$$

from (i) and (ii)

$$\int_a^b f\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$$

$$\int_a^b (f_1 + f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$$

Hence Proved.