THE RIEMANN-STIELTJES INTEGRAL

f is a constant function on $[a,b]$ defined by $f(x)=K$ and α is monotonically increasing on $[a,b]$ then show that $\int_a^b f\,d\alpha$ exists and

$$\int_a^b f\,d\alpha = K\{\alpha(b)-\alpha(a)\}$$

Proof

f is constant function

$$f(x)=K.$$

Now m_i and M_i are bounds of f in $[x_{i-1}, x_i]$ $1\le i\le n$.

$m_i = M_i = K$ where $P=\{a=x_0, x_1, x_2, \ldots, x_n=b\}$ is partition.

Now $L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

$$= k \sum_{i=1}^{n} \Delta\alpha_i$$

$$= k\left[\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + \dots + \alpha(x_{n-1}) - \alpha(x_{n-2}) + \alpha(x_n) - \alpha(x_{n-1})\right]$$

$$= k\left[\alpha(x_n) - \alpha(x_0)\right]$$

$$= k\left[\alpha(b) - \alpha(a)\right]$$

$$\underline{\int_a^b} f\,d\alpha = \sup\left[L(P,f,\alpha) : P \text{ is partition of } [a,b]\right]$$

$$\underline{\int_a^b} f\,d\alpha = K\left[\alpha(b)-\alpha(a)\right] \quad \text{—— (1)}$$

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i$$

$$= K\sum_{i=1}^{n} \Delta\alpha_i$$

$$= K\left[\alpha(x_n)-\alpha(x_0)\right]$$

$$= K\left[\alpha(b)-\alpha(a)\right]$$

$$\overline{\int_a^b} f\,d\alpha = \inf\left[U(P, f, \alpha) \quad P \text{ is partition of } [a,b]\right]$$

$$\overline{\int_a^b} f\,d\alpha = K\left[\alpha(b) - \alpha(a)\right] \quad \text{--- } (11)$$

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha \quad \left[\text{from } (1) \text{ and } (11)\right]$$

$$\Rightarrow \int_a^b f\,d\alpha \text{ exists.}$$

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

$$\int_a^b f\,d\alpha = k\left[\alpha(b)-\alpha(a)\right]$$

Hence Proved.