THE RIEMANN-STIELTJES INTEGRAL

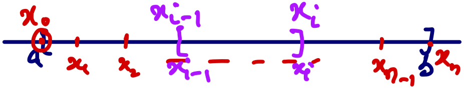
Thm Let $f:[a,b]\to R$ be a bounded function and $\alpha:[a,b]\to R$ be monotonically increasing function then for any partition P of $[a,b]$

$$m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M(\alpha(b)-\alpha(a))$$

where m, M are lower and upper Bounds of f in $[a,b]$.

Proof P is Partition of $[a,b]$

$$P = \{a = x_0, x_1, x_2 \text{-----}, x_n = b\}$$



$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$
$$M_i = \sup\{f(x) ; x \in [x_{i-1}, x_i]\}$$

$$m \leq m_i \leq M_i \leq M$$

$$m\,\Delta\alpha_i \leq m_i\,\Delta\alpha_i \leq M_i\,\Delta\alpha_i \leq M\,\Delta\alpha_i$$

$$m\sum_{i=1}^{n}\Delta\alpha_i \leq m_i \sum_{i=1}^{n}\Delta\alpha_i \leq M_i \sum_{i=1}^{n}\Delta\alpha_i \leq M \sum_{i=1}^{n}\Delta\alpha_i \quad \text{—} (1)$$

Now $\sum_{i=1}^{n}\Delta\alpha_i = \Delta\alpha_1 + \Delta\alpha_2 + \Delta\alpha_3 + \cdots + \Delta\alpha_{n-1} + \Delta\alpha_n$

$$= \alpha(\cancel{x_1}) - \alpha(x_0) + \alpha(\cancel{x_2}) - \alpha(\cancel{x_1}) + \alpha(x_3) - \alpha(\cancel{x_2}) +$$

$$------ \alpha(\cancel{x_{n-1}}) - \alpha(x_{n-2}) + \alpha(x_n) - \alpha(\cancel{x_{n-1}})$$

$$= \alpha(x_n) - \alpha(x_0)$$

$$\sum_{i=1}^{n} \Delta\alpha_i = \alpha(b) - \alpha(a)$$

Put $\sum_{i=1}^{n} \Delta\alpha_i = \alpha(b) - \alpha(a)$ in (1)

$$m[\alpha(b) - \alpha(a)] \leq L(P,f,\alpha) \leq U(P,f,\alpha) \leq M[\alpha(b) - \alpha(a)]$$

Hence Proved.