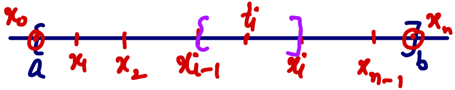
THE RIEMANN-STIELTJES INTEGRAL

$f \in R(\alpha)$ on $[a,b]$ then for $\epsilon > 0$ there exist a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[a,b]$ such that
$$\left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon$$
where t_i is any arbitrary point in $[x_{i-1}, x_i]$

Proof $\quad f \in R(\alpha)$

for $\epsilon > 0$ $\exists$ a partition P



s.t. $\quad P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \quad \text{--- (1)}$$

Let m_i and M_i are Bounds of $[x_{i-1}, x_i]$

$$\Rightarrow \quad m_i \leq f(t_i) \leq M_i \qquad [t_i \in [x_{i-1}, x_i]]$$

$$\Rightarrow \quad \sum_{i=1}^{n} m_i \,\Delta\alpha_i \leq \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \leq \sum_{i=1}^{n} M_i \,\Delta\alpha_i$$

$$\Rightarrow \quad L(P,f,\alpha) \leq \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \leq U(P,f,\alpha) \quad — (2)$$

Now $L(P,f,\alpha) \leq \int_a^b f\,d\alpha \leq U(P,f,\alpha)$ — (3)

[$f \in R(\alpha)$]

from (2) & (3)

$$\left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha \right| \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$f \in R(\alpha)$

$$\underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha$$

$$\overline{\int_a^b} f\,d\alpha = \inf[U(P,f,\alpha)] \leq U(P,f,\alpha)$$

$$\underline{\int_a^b} f\,d\alpha = \sup[L(P,f,\alpha)] \geq L(P,f,\alpha)$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon \qquad [\text{from } (1)]$$

Hence Proved.