THE RIEMANN-STIELTJES INTEGRAL

Theorem: Refinement Of Partition

If P^* is Refinement of P then

$$L(P^*, f, \alpha) \geq L(P, f, \alpha)$$

$$U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

Proof Let P is Partition of $[a, b]$

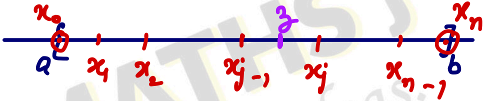
$$P = \{a = x_0, x_1 - - - - - - , x_{n-1}, x_n = b\}$$

m_i, M_i are Bounds of $[x_{i-1}, x_i]$ $1 \leq i \leq n$.

$$P' = \{a = x_0, x_1, x_2 - - - - - , x_{j-1}, z, x_j - - - - - x_{n-1}, x_n\}$$

$x_{j-1} < z < x_j$ $\quad$ $1 \le j \le n.$

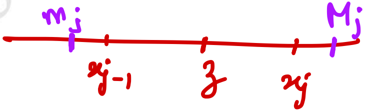
P' is the Refinement of P



Containing One Point more than P.

Let $s_1 = \inf\{f(x) : x \in (x_{j-1}, z)\}$

$r_1 = \sup\{f(x) : x \in (x_{j-1}, z)\}$



$r_2 = \sup\{f(x) : x \in (z, x_j)\}$

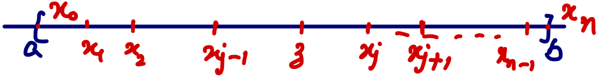
$s_2 = \inf\{f(x) : x \in (z, x_j)\}$

$s_1, s_2 \geq m_j$ $\qquad$ $r_1, r_2 \leq M_j$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \, \Delta\alpha_i$$

$$= \sum_{i=1}^{n} m_i \left[\alpha(x_i) - \alpha(x_{i-1})\right]$$

$$= \sum_{i=1}^{j-1} m_i \left(\alpha(x_i) - \alpha(x_{i-1})\right) + m_j \left(\alpha(x_j) - \alpha(x_{j-1})\right) + \sum_{i=j+1}^{n} m_i \left[\alpha(x_i) - \alpha(x_{i-1})\right]$$



$$= \sum_{i=1}^{j-1} m_i \left(\alpha(x_i) - \alpha(x_{i-1})\right) + m_j \left(\alpha(x_j) - \alpha(x_{j-1}) + \alpha(z) - \alpha(z)\right) + \sum_{i=j+1}^{n} m_i \left(\alpha(x_i) - \alpha(x_{i-1})\right)$$

$$= \sum_{i=1}^{j-1} m_i \left(\alpha(x_i) - \alpha(x_{i-1})\right) + m_j \left(\alpha(z) - \alpha(x_{j-1})\right) + m_j \left(\alpha(x_j) - \alpha(z)\right) + \sum_{i=j+1}^{n} m_i \left(\alpha(x_i) - \alpha(x_{i-1})\right)$$

$$\le \sum_{i=1}^{j-1} m_i \left(\alpha(x_i) - \alpha(x_{i-1})\right) + s_1 \left(\alpha(z) - \alpha(x_{j-1})\right) + s_2 \left(\alpha(x_j) - \alpha(z)\right) + \sum_{i=j+1}^{n} m_i \left(\alpha(x_i) - \alpha(x_{i-1})\right)$$

$$= L(P', f, \alpha)$$

$$L(P, f, \alpha) \leq L(P', f, \alpha)$$

IIIly $U(P, f, \alpha) \geq U(P', f, \alpha)$

If P^* contains k points more than P then by repeating the process we get

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

$$U(P, f, \alpha) \geq U(P^*, f, \alpha)$$

Hence Proved.