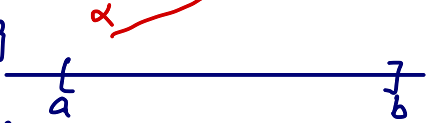
THE RIEMANN-STIELTJES INTEGRAL

If f is monotonic on $[a,b]$ and α is Continous and monotonically increasing on $[a,b]$ then $f \in R(\alpha)$ on $[a,b]$

Proof α is Continous and monotonically increasing.

$\therefore$ for any integer n $\exists$ a partition P s.t.

$P = \{a = x_0, x_1, x_2 \text{ ---- }, x_n = b\}$



$\Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$ $\quad i = 1, 2, 3 \ldots\ldots n.$ — (1)

f is monotonic function

$$\therefore \; m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1})$$

$$\therefore \; M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i)$$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \left[f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots \right.$$

$$\left. \cdots + f(x_n) - f(x_{n-1}) \right]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} [f(x_n) - f(x_0)]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)]$$

$< \epsilon$ if n is taken large enough

$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

$\therefore$ for given $\epsilon > 0$ $\exists$ a partition P of $[a, b]$ s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

$\Rightarrow$ $f \in R(\alpha)$ on $[a, b]$

Hence Proved.