THE RIEMANN-STIELTJES INTEGRAL

f is Continous on $[a,b]$ then $f \in R(\alpha)$ on $[a,b]$

Proof f is continous on $[a,b]$

$\Rightarrow$ f is uniformly continous on $[a,b]$

$\therefore$ By def. $\exists\ \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\epsilon}{\alpha(b) - \alpha(a)} \quad \forall\ x, y \in [a,b] \text{ when } |x-y| < \delta \quad \text{—(1)}$$

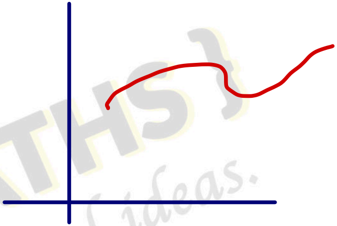
Now $P = \{a = x_0, x_1, \ldots, x_n = b\}$ is partition of $[a,b]$ with $\|P\| < \delta$

Now Since a Continous function on closed interval $[a,b]$ is Bounded and attains its Bounds.

$\therefore\ \exists\ c_i, d_i \in [x_{i-1}, x_i]$

s.t. $f(c_i) = m_i$

$f(d_i) = M_i$



where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$

$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$

$$|M_i - m_i| = |f(d_i) - f(c_i)| < \frac{\epsilon}{\alpha(b) - \alpha(a)} \quad \text{--- (2)}$$

[from (1)]

$|d_i - c_i| \leq |x_i - x_{i-1}| < \delta$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\, \Delta\alpha_i$$

$$< \frac{\epsilon}{\alpha(b)-\alpha(a)} \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \frac{\epsilon}{\alpha(b)-\alpha(a)} \left[\Delta\alpha_1 + \Delta\alpha_2 + \cdots + \Delta\alpha_n\right]$$

$$= \frac{\epsilon}{\alpha(b)-\alpha(a)} \left[\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + \cdots + \alpha(x_{n-1}) - \alpha(x_{n-2}) + \alpha(x_n) - \alpha(x_{n-1})\right]$$

$$= \frac{\epsilon}{\alpha(b)-\alpha(a)} \left[\alpha(x_n) - \alpha(x_0)\right]$$

$$= \frac{\epsilon}{\alpha(b)-\alpha(a)} \left(\alpha(b) - \alpha(a)\right) = \epsilon$$

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$f \in R(\alpha)$$

Hence Proved.