STATE AND PROOF DE MOIVRE'S THEOREM

State:- (i) In n is integer

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(ii) when n is fraction then one of value of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof:→ Case I: when n is +ve integer.

Let $n = 1$.

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta.$$

Let $n = 2$

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + (i\sin\theta)^2 + 2i\sin\theta\cos\theta.$$

$= \cos^2\theta + i^2 \sin^2\theta + 2i \sin\theta \cos\theta.$

$= \cos^2\theta - \sin^2\theta + i\,(2\sin\theta \cos\theta)$

$= \cos 2\theta + i \sin 2\theta$ $\left[\begin{array}{l} \cos^2\theta - \sin^2\theta = \cos 2\theta \\ 2\sin\theta \cos\theta = \sin 2\theta \end{array}\right]$

$(\cos\theta + i \sin\theta)^2 = \cos 2\theta + i \sin 2\theta.$

Result is true for $n = 2.$

$\underline{\underline{n = 3.}}$ $(\cos\theta + i\sin\theta)^3 = (\cos\theta + i \sin\theta)^2 (\cos\theta + i\sin\theta)$

$= (\cos 2\theta + i \sin 2\theta)(\cos\theta + i\sin\theta)$

$= \cos 2\theta \cos\theta + i\,(\cos 2\theta \sin\theta + \sin 2\theta \cos\theta) + i^2 \sin 2\theta \sin\theta.$

$\boxed{i^2 = -1}$

$$= (\cos 2\theta \cos\theta - \sin 2\theta \sin\theta) + i(\cos 2\theta \sin\theta + \sin 2\theta \cos\theta)$$

$$= \cos(2\theta + \theta) + i(\sin(2\theta + \theta))$$

$$= \cos 3\theta + i \sin 3\theta.$$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i \sin 3\theta.$$

Result is true for $n=3$.

By Repeating the same process.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta. \quad \text{--- } (1)$$

Case-II n is −ve integer.

$$(\cos\theta + i\sin\theta)^{-n} = \frac{1}{(\cos\theta + i\sin\theta)^{n}}$$

$$= \frac{1}{\cos n\theta + i\sin n\theta} \quad [\text{By Case I}]$$

$$= \frac{(\cos n\theta - i\sin n\theta)}{(\cos n\theta + i\sin n\theta)(\cos\theta - i\sin n\theta)}$$

$$= \frac{\cos n\theta - i\sin n\theta.}{\cos^2 n\theta - i^2\sin^2 n\theta.}$$

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$$

$$= \cos n\theta - i\sin n\theta.$$

$$(\cos\theta + i\sin n\theta)^{-n} = \cos n\theta - i\sin n\theta \qquad \left[\begin{array}{l}\cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta\end{array}\right]$$

$$= \cos(-n\theta) + i\sin(-n\theta)$$

Result is true when n is $-ve$. integer. — (ii)

Case III $n = 0$

$$(\cos\theta + i\sin\theta)^0 = 1. = \cos(0.\theta) + i\sin(0.\theta)$$

The Result is true for $n=0$. — (iii)

By (i), (ii) and (iii).

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all integers

Case IV when n is fraction.

Let $n = p/q$ where $q > 0$ and $p > 0$

$p < 0$

Now $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\frac{q\theta}{q} + i\sin\frac{q\theta}{q}$

[By case I]

$= \cos\theta + i\sin\theta.$

Take q^{th} Root on both sides.

$$\left[\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{2}\right]^{1/2} = (\cos\theta + i\sin\theta)^{1/2}$$

$$\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} = (\cos\theta + i\sin\theta)^{1/2} \quad \text{— (iv).}$$

$$(\cos\theta + i\sin\theta)^{p/2} = \left[(\cos\theta + i\sin\theta)^{p}\right]^{1/2}$$

$$= (\cos p\theta + i\sin p\theta)^{1/2}$$

$$= \cos\frac{p}{2}\theta + i\sin\frac{p}{2}\theta \quad [\text{from (iv)}].$$

$$\Rightarrow (\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^{n}$$

when n is fraction. Hence proved.