

# STATE AND PROOF DE MOIVRE'S THEOREM

State:- (i) In  $n$  is integer

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

(ii) when  $n$  is fraction then one of value of

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

Proof:- Case I:- When  $n$  is +ve integer.

Let  $n=1$ .

$$(\cos \theta + i \sin \theta)^1 = \cos \theta + i \sin \theta.$$

Let  $n=2$

$$(\cos \theta + i \sin \theta)^2 = \cos^2 \theta + (i \sin \theta)^2 + 2i \sin \theta \cos \theta.$$

$$= \cos^2 \theta + i^2 \sin^2 \theta + 2i \sin \theta \cos \theta.$$

$$= \cos^2 \theta - \sin^2 \theta + i (2 \sin \theta \cos \theta)$$

$$= \cos 2\theta + i \sin 2\theta$$

$$\left[ \begin{array}{l} \cos^2 \theta - \sin^2 \theta = \cos 2\theta \\ 2 \sin \theta \cos \theta = \sin 2\theta \end{array} \right]$$

$$(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta.$$

Result is true for  $n=2$ .

$$\underline{\underline{n=3.}} \quad (\cos \theta + i \sin \theta)^3 = (\cos \theta + i \sin \theta)^2 (\cos \theta + i \sin \theta)$$

$$= (\cos 2\theta + i \sin 2\theta) (\cos \theta + i \sin \theta)$$

$$= \cos 2\theta \cos \theta + i (\cos 2\theta \sin \theta + \sin 2\theta \cos \theta) +$$

$$i^2 \sin 2\theta \sin \theta.$$

$$\boxed{i^2 = -1}$$

$$\begin{aligned} &= (\cos 2\theta \cos \theta - \sin 2\theta \sin \theta) + i (\cos 2\theta \sin \theta + \sin 2\theta \cos \theta) \\ &= \cos (2\theta + \theta) + i (\sin (2\theta + \theta)) \\ &= \cos 3\theta + i \sin 3\theta. \end{aligned}$$

$$(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta.$$

Result is true for  $n=3$ .

By Repeating the same process

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \quad \text{--- (1)}$$

Case-II  $n$  is -ve integer.

$$(\cos\theta + i\sin\theta)^{-n} = \frac{1}{(\cos\theta + i\sin\theta)^n}$$

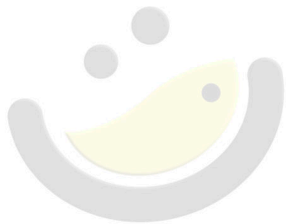
$$= \frac{1}{\cos n\theta + i\sin n\theta} \quad \left[ \text{By Case I} \right]$$

$$= \frac{(\cos n\theta - i\sin n\theta)}{(\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta)}$$

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta - i^2 \sin^2 n\theta}$$

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta - i^2 \sin^2 n\theta}$$

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta - i^2 \sin^2 n\theta}$$



$$= \frac{\cos n\theta - i \sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$$

$$= \cos n\theta - i \sin n\theta$$

$$\begin{aligned} (\cos\theta + i \sin\theta)^{-n} &= \cos n\theta - i \sin n\theta \\ &= \cos(-n\theta) + i \sin(-n\theta) \end{aligned} \begin{cases} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{cases}$$

Result is true when  $n$  is -ve. integer. - (ii)

Case III

$$\underline{\underline{n=0}}$$

$$(\cos\theta + i \sin\theta)^0 = 1 = \cos(0 \cdot \theta) + i \sin(0 \cdot \theta)$$

The Result is true for  $n=0$ . - (iii)

By (i), (ii) and (iii).

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \text{ for all integers } n$$

Case IV

When  $n$  is fraction.

$$\text{Let } n = \frac{p}{q} \text{ where } q > 0 \text{ and } p > 0$$

$$p < 0$$

Now

$$\left( \cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right)^q = \cos \frac{q\theta}{q} + i \sin \frac{q\theta}{q}$$

[By Case I]

$$= \cos \theta + i \sin \theta.$$

Take  $q^{\text{th}}$  Root on both sides.

$$\left[ \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)^2 \right]^{1/2} = (\cos \theta + i \sin \theta)^{1/2}$$

$$\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} = (\cos \theta + i \sin \theta)^{1/2} \quad \text{--- (17)}$$

$$(\cos \theta + i \sin \theta)^{p/2} = [(\cos \theta + i \sin \theta)^p]^{1/2}$$

$$= (\cos p\theta + i \sin p\theta)^{1/2}$$

$$= \cos \frac{p\theta}{2} + i \sin \frac{p\theta}{2} \quad \text{[from (17)]}$$

$$\Rightarrow (\cos n\theta + i \sin n\theta) = (\cos \theta + i \sin \theta)^n$$

When  $n$  is fraction. Hence proved.