THE RIEMANN-STIELTJES INTEGRAL

Condition Of Integrability

Let f be a bounded and α be a monotonically increasing functions on $[a,b]$ then $f \in R(\alpha)$ iff for $\epsilon > 0$ there exist a partition P of $[a,b]$ s.t.

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

Proof

$f \in R(\alpha)$ on $[a,b]$

So $$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha \quad \text{--- } (1)$$

$\because \overline{\int_a^b} f\,d\alpha = \inf\{U(P,f,\alpha) : P \text{ is partition of } [a,b]\}$

for given $\epsilon > 0$ $\exists$ a partition P_1 of $[a,b]$ s.t.

$$U(P_1, f, \alpha) < \overline{\int_a^b} f\,d\alpha + \epsilon/2$$

$$= \int_a^b f\,d\alpha + \epsilon/2 \quad [\text{from } (1)] \quad \text{—} (2)$$

$\underline{\int_a^b} f\,d\alpha = \sup\{L(P,f,\alpha) : P \text{ is partition of } [a,b]\}$

for $\epsilon > 0$ $\exists$ partition P_2 of $[a,b]$

$$L(P_2, f, \alpha) > \int_a^b f\,d\alpha - \epsilon/2$$

$$\Rightarrow L(P_2, f, \alpha) > \int_a^b f\,d\alpha - \epsilon/2 \quad — ③$$

[from ①]

Let $P = P_1 \cup P_2$ is Common Refinement

$$U(P, f, \alpha) < \int_a^b f\,d\alpha + \epsilon/2 \quad —(i) \qquad [\text{from } ②]$$

$$L(P, f, \alpha) > \int_a^b f\,d\alpha - \epsilon/2 \qquad [\text{from } ③]$$

$$-L(P, f, \alpha) < -\int_a^b f\,d\alpha + \epsilon/2 \quad —(ii)$$

Add (i) & (ii)

$$U(P, f, \alpha) - L(P, f, \alpha) = \int_a^b f\,d\alpha + \epsilon/2 + \left[-\int_a^b f\,d\alpha\right] + \epsilon/2$$

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

<u>Converse</u> $\quad U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

<u>To prove</u> $\quad f \in R(\alpha)$ on $[a, b]$

$$\underline{\int_a^b} f\,d\alpha \geq L(P, f, \alpha)$$

$$\overline{\int_a^b} f\,d\alpha \leq U(P, f, \alpha) \quad \text{(iii)}$$

$$-\underline{\int_a^b} f\,d\alpha \leq -L(P,f,\alpha) \quad \text{---} (iv)$$

Add (iii) & (iv)

$$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$0 \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon \qquad \left[\because \overline{\int_a^b} f\,d\alpha \geq \underline{\int_a^b} f\,d\alpha\right]$$

But $\epsilon > 0$ may be arbitrary small.

$$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha = 0$$

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

$$\Rightarrow \quad f \in R(\alpha) \text{ on } [a,b]$$

Hence proved.