

THE RIEMANN-STIELTJES INTEGRAL

Condition Of Integrability

Let f be a bounded and α be a monotonically increasing functions on $[a, b]$ then $f \in R(\alpha)$ iff for $\epsilon > 0$ there exist a partition P of $[a, b]$ s.t.

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Proof

$f \in R(\alpha)$ on $[a, b]$

So

$$\int_a^b f d\alpha = \int_a^b f d\alpha = \int_a^b f d\alpha \quad \text{--- (1)}$$

$$\therefore \int_a^b f d\alpha = \inf \{ U(P, f, \alpha) : P \text{ is partition of } [a, b] \}$$

for given $\epsilon > 0 \exists$ a partition P_1 of $[a, b]$ s.t.

$$U(P_1, f, \alpha) < \int_a^b f d\alpha + \epsilon/2$$
$$= \int_a^b f d\alpha + \epsilon/2 \quad [\text{from (1)}] \quad - (2)$$

$$\int_a^b f d\alpha = \sup \{ L(P, f, \alpha) : P \text{ is partition of } [a, b] \}$$

for $\epsilon > 0 \exists$ partition P_2 of $[a, b]$

$$L(P_2, f, \alpha) > \int_a^b f d\alpha - \epsilon/2$$

$$\Rightarrow L(P_2, f, \alpha) > \int_a^b f d\alpha - \epsilon/2 \quad \text{--- (3)}$$

[from (1)]

Let $P = P_1 \cup P_2$ is Common Refinement

$$U(P, f, \alpha) < \int_a^b f d\alpha + \epsilon/2 \quad \text{--- (i)} \quad \text{[from (2)]}$$

$$L(P, f, \alpha) > \int_a^b f d\alpha - \epsilon/2 \quad \text{[from (3)]}$$

$$- L(P, f, \alpha) < - \int_a^b f d\alpha + \epsilon/2 \quad \text{--- (ii)}$$

Add (i) + (ii)

$$U(P, f, \alpha) - L(P, f, \alpha) = \int_a^b f d\alpha + \epsilon_2 + \left[- \int_a^b f d\alpha \right] + \epsilon_2$$

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Converse

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

To prove

$f \in R(\alpha)$ on $[a, b]$

$$\int_a^b f d\alpha \Rightarrow L(P, f, \alpha)$$

$$\int_a^b f d\alpha \leq U(P, f, \alpha) \quad (\text{iii})$$

$$-\int_a^b f dx \leq -L(P, f, \alpha) \quad \text{--- (iv)}$$

Add (iii) & (iv)

$$\int_a^b f dx - \int_a^b f dx \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

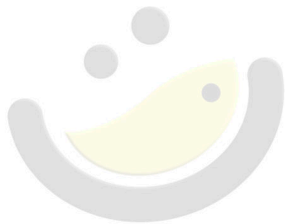
$$0 \leq \int_a^b f dx - \int_a^b f dx < \epsilon \quad \left[\because \int_a^b f dx \geq \int_a^b f dx \right]$$

But $\epsilon > 0$ may be arbitrary small.

$$\int_a^b f dx - \int_a^b f dx = 0$$

$$\int_a^b f d\alpha = \int_a^b f d\alpha$$

$\Rightarrow f \in R(\alpha)$ on $[a, b]$
Hence proved.



OMG! MATHS }
The poetry of logical ideas.