THE RIEMANN-STIELTJES INTEGRAL

Let $f \in R(\alpha)$ on $[a,b]$ and $m \leq f(x) \leq M \ \forall x \in [a,b]$

Let $\phi : [m,M] \to R$ be a continous function. Then

$h(x) = \phi \circ f \in R(\alpha)$ on $[a,b]$.

Proof

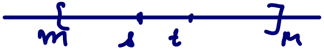
$\phi : [m,M] \to R$ is continous.

$\Rightarrow \phi$ is uniformly continous on $[a,b]$.

$\Rightarrow \exists \ \delta > 0$ for given $\epsilon > 0$ s.t.

$|\phi(s) - \phi(t)| < \epsilon \quad \forall \ s,t \in [m,M]$ where $|s-t| < \delta$ — (1)

Let $\boxed{\delta < \epsilon}$



$f \in R(\alpha)$ on $[a,b]$

$\therefore$ for $\delta > 0$ $\exists$ a partition P

$$P = \{a = x_0, x_1, x_2 \, ------, x_n = b\}$$

s.t. $\quad U(P,f,\alpha) - L(P,f,\alpha) < \delta \quad — ②$

Now Let $\quad m_i = \operatorname{Inf} [f(x) : x \in [x_{i-1}, x_i]]$

$$M_i = \operatorname{Sup} [f(x) : x \in [x_{i-1}, x_i]]$$

$$m_i' = \operatorname{Inf} [h(x) : x \in [x_{i-1}, x_i]]$$

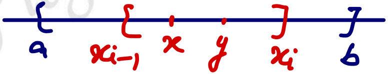
$$M_i' = \operatorname{Sup} [h(x) : x \in [x_{i-1}, x_i]]$$

$$A = \{ i : M_i - m_i < \delta \}$$

$$B = \{ i : M_i - m_i \geq \delta \}$$

Now when $i \in A$ and

$$x_{i-1} < x < y < x_i$$



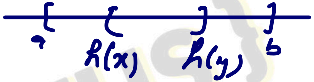
$$|f(x) - f(y)| \leq M_i - m_i < \delta$$

$$|f(x) - f(y)| < \delta$$

$$\Rightarrow \phi(f(x)) - \phi(f(y)) < \epsilon \quad [\text{from } (1)]$$

$$\Rightarrow |h(x) - h(y)| < \epsilon$$

$\Rightarrow |M_i' - m_i'| < \epsilon$ when $i \in A$.



$$\sum_{i \in A} (M_i' - m_i') \Delta\alpha_i < \epsilon \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \epsilon \left[\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + \text{- - -} \right.$$
$$\left. + \text{- - -} + \alpha(x_n) - \alpha(x_{n-1}) \right]$$

$$= \epsilon \left[\alpha(x_n) - \alpha(x_0) \right]$$

$$= \epsilon \left[\alpha(b) - \alpha(a) \right] \quad \text{— (iii)}$$

When $i \in B$.

$$\sum_{i \in B} \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i)\,\Delta\alpha_i$$

$$= U(P, f, \alpha) - L(P, f, \alpha) < \delta < \epsilon \quad [\text{from } (2)]$$

When $i \in B$.

$$M_i' - m_i' = |M_i - m_i| \leq |M_i'| + |m_i'|$$

$$\leq k + k = 2k \qquad \text{— (IV)}$$

where $|h(x)| \leq k$

$\forall x \in [a, b]$.

$$U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i} (M_i' - m_i')\,\Delta\alpha_i$$

$$= \sum_{i \in A} (M_i' - m_i')\,\Delta\alpha_i + \sum_{i \in B} (M_i' - m_i')\,\Delta\alpha_i$$

$$< \epsilon\,[\alpha(b) - \alpha(a)] + 2K\epsilon \qquad [\text{from } (11) \text{ and } (12)]$$

$$= \epsilon\,[\alpha(b) - \alpha(a) + 2K] = \epsilon'$$

$$U(P, h, \alpha) - L(P, h, \alpha) < \epsilon'$$

$\Rightarrow$ $h \in R(\alpha)$ on $[a, b]$.

Hence Proved.