THE RIEMANN-STIELTJES INTEGRAL

Let f be bounded on $[a,b]$ which has finitely many points of discontinuity in $[a,b]$ and α is monotonically increasing function which is continous at all those points where f is discontinous. Then $f \in R(\alpha)$ on $[a,b]$

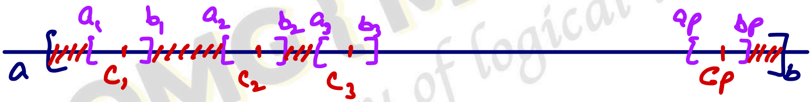
Proof

f is discontinous at finitely many points on $[a,b]$

Let $E = \{c_1, c_2, \_\_\_\_, c_p\}$ are points where f is discontinous.

Now α is Continous for every points of E {Given}

$\therefore$ We Can Cover each point of E with an closed interval.



$c_1 \in [a_1, b_1]$

$c_2 \in [a_2, b_2]$

$\vdots$

$c_p \in [a_p, b_p]$ also $\sum_{i=1}^{p} \alpha(b_i) - \alpha(a_i) < \epsilon$

$R = [[a_1, b_1] ; [a_2 b_2] \text{ ------ } [a_p, b_p]]$ — (1)

Now f is Continous in each of $p+1$ intervals.

$K = \{[a, a_1] ; [b_1, a_2] ; [b_2, a_3] \text{ ------ } [b_p, b]\}$ — (2)

Let $P_1, P_2 \text{ ---- } P_{p+1}$ are partition of intervals of K.

Now $U(P_j, f, \alpha) - L(P_j, f, \alpha) < \epsilon \quad \forall \ 1 \leq j \leq p+1.$ — (3)

[Continous function on interval $[a, b]$ is R-S integrable on $[a, b]$]

$[a, b]$ is divided into two groups

$[a, b] = R + K$

Now each interval of R is sub-interval of $[a, b]$
so oscillation of f in each interval $\leqslant M - m$.

Let P_1' is partition of $[a_1, b_1]$
P_2' is partition of $[a_2, b_2]$
$\vdots$
P_p' is partition of $[a_p, b_p]$

$$U(P_i', f, \alpha) - L(P_i', f, \alpha) = \sum_{i=1}^{p} (M_i - m_i)\,\Delta\alpha_i < (M - m)\cancel{\epsilon'}\,\epsilon$$

$$U(P_i', f, \alpha) - L(P_i', f, \alpha) < \epsilon' \quad \text{--- (4)}$$

Now Let P is Common Partition of $[a, b]$

$$P = P_j \cup P_i' \qquad 1 \leq j \leq p+1$$
$$1 \leq i \leq p$$

$$U(P, f, \alpha) - L(P, f, \alpha) = U(P_j, f, \alpha) - L(P_j, f, \alpha)$$
$$+ U(P_i', f, \alpha) - L(P_i', f, \alpha)$$
$$< \epsilon + \epsilon' = \epsilon''$$

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon''$$

Hence $f \in R(\alpha)$ on $[a, b]$.