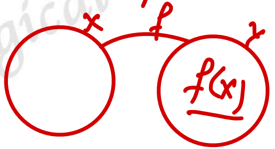
Continuity and Compactness

X and Y are metric spaces and $f : X \to Y$ be a continous function if

X is compact then $f(X)$ is also compact.



Proof Let $\{V_\alpha : \alpha \in \Lambda\}$ is open cover of $f(X)$

$$f(X) \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$$

$$\Rightarrow X \subseteq f^{-1}\left(\bigcup_{\alpha \in \Lambda} (V_\alpha)\right)$$

$$\Rightarrow X \subseteq \bigcup_{\alpha \in \Lambda} \left(f^{-1}(V_\alpha)\right) \quad \text{— (1)}$$

V_α is open in Y so $f^{-1}(V_\alpha)$ is open in X. [$\because f$ is continous]

— (ii)

from (i) and (ii)

$f^{-1}(V_\alpha)$ is open cover of X

But X is compact.

$\Rightarrow \exists$ a finite subcover s.t.

$\{f^{-1}(V_{\alpha_i}) \quad 1 \leq i \leq n\}$ is finite subcover of X.

$$X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$$

$$f(X) \subseteq f\left[\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right]$$

$$= \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i}))$$

$$= \bigcup_{i=1}^{n} V_{\alpha_i}$$

$$f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$$

$\Rightarrow$ $\{V_{\alpha_i} : 1 \leq i \leq n\}$ is finite subcover of $f(X)$

$\Rightarrow$ $f(X)$ is compact.