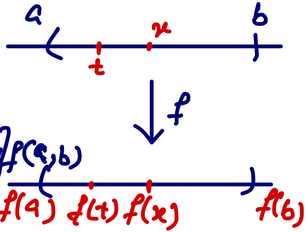
Limit and Continuity

f is monotonically increasing function defined on (a,b) then $f(x^+)$ and $f(x^-)$ exists at every point x of (a,b) and

$$\sup_{a<t<x} f(t) = f(x^-) \le f(x) \le f(x^+) = \inf_{x<t<b} f(t)$$

Proof

$A = \{f(t) : a < t < x\}$



$f(t) \le f(x)$ [$\because f$ is monotonically increasing]

$\Rightarrow$ set A is Bounded above

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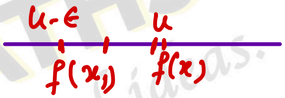
By Order Completeness property.

$\sup A$ exists

Let $\sup A = u$.

By def of sup.

for $\epsilon > 0$ $\exists$ $x_1 \in (a, x)$



s.t. $u-\epsilon < f(x_1) \leq u \quad \forall \; x_1 \in (a, x)$

$u-\epsilon < f(t) \leq u \quad \forall \; t \; (x_1, x)$

$u-\epsilon < f(t) < u+\epsilon \quad \forall \; t \, (x_1, x)$ $\quad \delta > 0$, $x_1 - x = \delta$

$u-\epsilon < f(t) < u+\epsilon$ for $x-\delta < t < x$

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$-\epsilon < f(t) - u < \epsilon \quad$ for $\quad x-\delta < t < x$

$|f(t) - u| < \epsilon \quad$ for $\quad x-\delta < t < x.$

$f(x^-) = u.$

$u = \sup A$

$A =$ upper bound $\underline{f(x)}$

$u \leq f(x)$

$$\sup_{a<t<x} f(t) = f(x^-) \leq f(x) \quad -\ (1)$$

$B = \{ f(t) \quad x < t < b \}$

Now $f(t) \geq f(x)$

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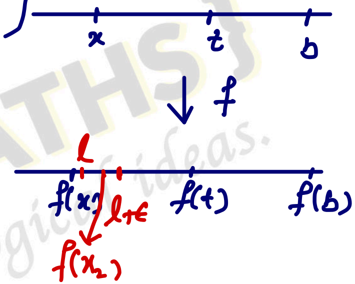
[$\because f$ is Monotonically increasing]



Set B is Bounded Below

By order Completeness property.

Inf B exists

Let Inf $B = l$.

By def. of Infimum.

for $\epsilon > 0$ $\exists$ $x_2 \in (x, b)$

$$l \leq f(x_2) < l + \epsilon$$

$$l \leq f(t) < l + \epsilon \quad \forall\, t \in (x, x_2)$$

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$$l-\epsilon < f(t) < l+\epsilon \quad \text{for } t \in (x, x_2)$$

$$x_2 - x = \delta$$

$$l-\epsilon < f(t) < l+\epsilon \quad \text{for } x < t < x+\delta$$

$$|f(t) - l| < \epsilon \quad \text{for } x < t < x+\delta$$

$$\Rightarrow \quad f(x^+) = l$$

Now $\quad l \geq f(x) \quad$ [$\because$ l is $\inf B$ also B has lowerBound $f(x)$]

$$f(x^+) \geq f(x)$$

$$\inf_{x<t<b} f(t) = f(x^+) \geq f(x) \quad \text{--- } ②$$

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

Hence Proved.

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