Supremum and Infimum

$C = \{a+b \quad a \in A,\ b \in B\}$ A and B are Bounded then show that.

$$\text{Sup } C = \text{Sup } A + \text{Sup } B.$$

$$\text{Inf } C = \text{Inf } A + \text{Inf } B.$$

Proof A and B are Bounded sets.

$\therefore$ A and B are Bounded above

By order Completeness property.

Supremum of A and B exist.

Let $\text{Sup } A = \alpha \qquad \text{Sup } B = \beta$

$\alpha \geq a \quad \forall\, a \in A$

$\beta \geq b \quad \forall\, b \in B$

$\alpha + \beta \geq a + b$ where $\underline{a+b \in C}$.

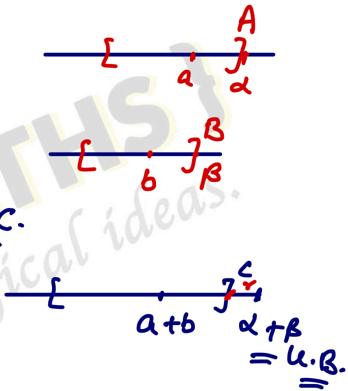
$\Rightarrow$ $(\alpha+\beta)$ is upper of C.

$\Rightarrow$ C is Bounded above.

By order Completeness property.

sup C exists

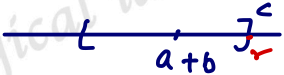
Let $\sup C = \gamma$. $=$ l.u.B of C.



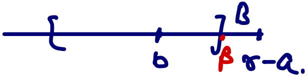
$\Rightarrow \quad \gamma \leq \alpha + \beta$

$$\sup C \leq \sup A + \sup B \quad \text{—} (1)$$

__Now__ $\quad a + b \leq \gamma$

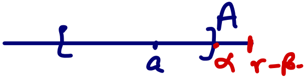


$\Rightarrow \quad b \leq \gamma - a$ = u.B of B.



$\Rightarrow \quad \beta \leq \gamma - a.$

$\Rightarrow \quad \beta + a \leq \gamma$



$\Rightarrow \quad a \leq \gamma - \beta$ = u.B of A

$\Rightarrow \quad \alpha \leq \gamma - \beta$

$\Rightarrow \quad \alpha + \beta \leq \gamma$

$\Rightarrow \quad \sup A + \sup B \leq \sup C \quad$ — (ii)

from (i) & (ii)

$\sup A + \sup B = \sup C.$