Limit of a function

Let (X, d_1) and (Y, d_2) be two metric spaces.

Let $E \subseteq X$ and c be limit point of E

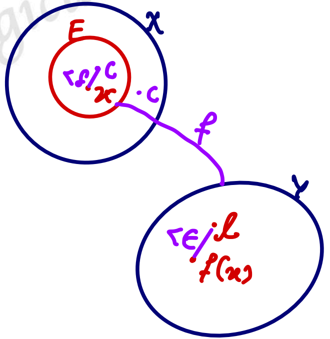
Let $f : E \to Y$ be a function.

$$\lim_{x \to c} f(x) = l \qquad [\, l \in Y \,]$$

if for given $\epsilon > 0$ $\exists$ $\delta > 0$

s.t. $d_2(f(x), l) < \epsilon$

when $d_1(x, c) < \delta$ $\boxed{x \in E}$



$$E \cap (c-\delta, c+\delta) - \{c\} \neq \phi.$$

Thm Let (X, d_1) and (Y, d_2) be two metric spaces. Let $E \subseteq X$, c is limit of E, $f: E \to Y$ then $\lim\limits_{x \to c} f(x)$ if exist is unique.

Proof:

Let $\lim_{x \to c} f(x) = l_1$

$\lim_{x \to c} f(x) = l_2$

By def. of limit of function.

for $\epsilon > 0$ $\exists$ $\delta_1, \delta_2 > 0$ s.t.

$d_2(f(x), l_1) < \epsilon/2$ when $d_1(x, c) < \delta_1$

$d_2(f(x), l_2) < \epsilon/2$ when $d_1(x, c) < \delta_2$

$\delta = \text{Min}\{\delta_1, \delta_2\}$

$$\left.\begin{aligned} d_2\left(f(x), l_1\right) &< \epsilon/2 \\ d_2\left(f(x), l_2\right) &< \epsilon/2 \end{aligned}\right\} \text{ [when } d_1(x, c) < \delta \text{]}$$

$$d(l_1, l_2) \leq d_2\left(f(x), l_1\right) + d_2\left(f(x), l_2\right)$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \text{ [when } d_1(x, c) < \delta \text{]}$$

$d(l_1, l_2) < \epsilon$

ϵ is very small

$d(l_1, l_2) = 0$

$l_1 = l_2$. $\Rightarrow \lim\limits_{x \to c} f(x)$ is unique.