Limit Inferior and Limit Superior

Let $\{x_n\}$ is a Bounded sequence in $\mathbb{R}$ then.

(a) $\overline{\lim}\; x_n = \inf_{n} \sup_{m \geq n} x_m.$

(b) $\underline{\lim}\; x_n = \sup_{n} \inf_{m \geq n} x_m.$

Proof

$y_n = \sup_{m \geq n} x_m = \sup\{x_n, x_{n+1} - - - - - \}$

$y_{n+1} = \sup\{x_{n+1}, x_{n+2}, x_{n+3} - - - - - \}$

$y_{n+1} < y_n \quad \forall\, n$

y_n is Monotonically Decreasing sequence
Also y_n is Bounded Below [$\because \{x_n\}$ is Bounded]

Every monotonically decreasing and Bounded Below sequence converges to its infimum.

$$\{y_n\} \longrightarrow y \quad \Rightarrow y = \inf_n y_n$$

By def of Convergence.
for $\epsilon > 0$ $\exists\, n_0 \in \mathbb{N}$
s.t. $|y_n - y| < \epsilon \quad \forall\, n \geq n_0$

$$-\epsilon < y_n - y < \epsilon \quad \forall\, n \geq n_0$$

$$y-\epsilon < y_n < y+\epsilon \quad \forall\ n \geq n_0$$

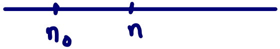
$$y_n > y-\epsilon \quad \forall\ n \geq n_0$$

$$\sup_{m \geq n} x_m > y-\epsilon \quad \forall\ n \geq n_0$$

$x_n > y-\epsilon$ for infinite many n.

$$y_n < y+\epsilon \quad \forall\ n \geq n_0$$

$$\sup_{m \geq n} x_m < y+\epsilon \quad \forall\ n \geq n_0$$



$x_n < y+\epsilon$ for infinite many n.

$x_n > y+\epsilon$ for atmost finite terms.

$\overline{\lim}\, x_n = x$ iff

(a) $x_n > x + \epsilon$ for atmost finite terms.

(b) $x_n > x - \epsilon$ for infinite many n.

$\Rightarrow \overline{\lim}\, x_n = y. = \inf\limits_{n} y_n = \inf\limits_{n} \left[\sup\limits_{m \geq n} x_m \right]$

$\Rightarrow \overline{\lim}\, x_n = \inf\limits_{n} \left[\sup\limits_{m \geq n} x_m \right]$