

Limit Inferior and Limit Superior

$\{x_n\}$ and $\{y_n\}$ are Bounded sequence in \mathbb{R} then.

$$\begin{aligned}\underline{\lim} x_n + \underline{\lim} y_n &\leq \underline{\lim} (x_n + y_n) \leq \overline{\lim} (x_n + y_n) \\ &\leq \overline{\lim} x_n + \overline{\lim} y_n.\end{aligned}$$

Proof

$$\underline{\lim} x_n = \sup_n \left(\inf_{m \geq n} x_m \right)$$

$$\underline{\lim} y_n = \sup_n \left(\inf_{m \geq n} y_m \right)$$

$$s_n = \inf_{m \geq n} x_m$$

$$s_n \leq x_m \quad \forall m \geq n$$

$$t_n = \inf_{m \geq n} y_m$$

$$t_n \leq y_m \quad \forall m \geq n.$$

$$S_n + t_n \leq x_m + y_m \quad \forall m \geq n$$

$$S_n + t_n \leq \inf_{m \geq n} (x_m + y_m) \quad \text{--- ①}$$

$$\underline{\lim} x_n + \underline{\lim} y_n = \sup_n \left[\inf_{m \geq n} x_m \right] + \sup_n \left[\inf_{m \geq n} y_m \right]$$

$$= \sup_n [S_n] + \sup_n [t_n]$$

$$= \lim_{n \rightarrow \infty} S_n + \lim_{n \rightarrow \infty} t_n$$

$$= \lim_{n \rightarrow \infty} (S_n + t_n)$$

$$\leq \lim_{n \rightarrow \infty} \inf_{m \geq n} (x_m + y_m) \quad [\text{from ①}]$$



$$= \sup_n \left[\inf_{m \geq n} (x_m + y_m) \right]$$

$$= \underline{\lim} (x_n + y_n)$$

$$\underline{\lim} x_n + \underline{\lim} y_n \leq \underline{\lim} (x_n + y_n) \quad \text{--- (2)}$$

Now $\overline{\lim} x_n = \inf_n \left[\sup_{m \geq n} x_m \right]$

$$\overline{\lim} y_n = \inf_n \left[\sup_{m \geq n} y_m \right]$$

$$p_n = \sup_{m \geq n} x_m \quad \text{and} \quad q_n = \sup_{m \geq n} y_m$$

$$p_n \geq x_m \quad \forall m \geq n \quad \text{and} \quad q_n \geq y_m \quad \forall m \geq n$$

$$p_n + q_n \geq x_m + y_m \quad \forall m \geq n.$$

$$p_n + q_n \geq \sup_{m \geq n} (x_m + y_m) \quad \text{--- (3)}$$

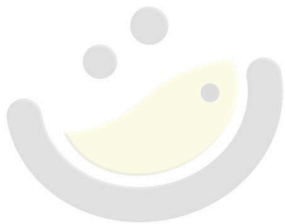
$$\overline{\lim} x_n + \overline{\lim} y_n = \inf_n \left[\sup_{m \geq n} x_m \right] + \inf_n \left[\sup_{m \geq n} y_m \right]$$

$$= \inf_n [p_n] + \inf_n [q_n]$$

$$= \lim_{n \rightarrow \infty} p_n + \lim_{n \rightarrow \infty} q_n$$

$$= \lim_{n \rightarrow \infty} (p_n + q_n)$$

$$\geq \sup_{m \geq n} (x_m + y_m) \quad [\text{from (3)}]$$



$$\begin{aligned}
 &= \inf_n \left[\sup_{m \geq n} (x_m + y_m) \right] \\
 &= \overline{\lim} (x_n + y_n) \\
 \overline{\lim} x_n + \overline{\lim} y_n &\geq \overline{\lim} (x_n + y_n) \quad - (4)
 \end{aligned}$$

By def

$$\underline{\lim} (x_n + y_n) \leq \overline{\lim} (x_n + y_n) \quad - (5)$$

$$\underline{\lim} x_n + \underline{\lim} y_n \leq \underline{\lim} (x_n + y_n) \leq \overline{\lim} (x_n + y_n)$$

$$\leq \overline{\lim} x_n + \overline{\lim} y_n \quad \left[\text{from (2), (4)} \right]$$

Hence proved.

and (5)