Limit Inferior and Limit Superior

$\{x_n\}$ and $\{y_n\}$ are Bounded sequence in $\mathbb{R}$ then.

$$\underline{\lim}\, x_n + \underline{\lim}\, y_n \leq \underline{\lim}\,(x_n+y_n) \leq \overline{\lim}\,(x_n+y_n)$$
$$\leq \overline{\lim}\, x_n + \overline{\lim}\, y_n.$$

Proof

$$\underline{\lim}\, x_n = \sup_n \left(\inf_{m\geq n} x_m \right)$$

$$\underline{\lim}\, y_n = \sup_n \left(\inf_{m\geq n} y_m \right)$$

$$s_n = \inf_{m\geq n} x_m \qquad t_n = \inf_{m\geq n} y_m$$

$$s_n \leq x_m \ \forall\ m\geq n \qquad t_n \leq y_m \ \forall\ m\geq n.$$

$$s_n + t_n \leq x_m + y_m \quad \forall\ m \geq n$$

$$s_n + t_n \leq \inf_{m \geq n} (x_m + y_m) \quad \text{— (1)}$$

$$\underline{\lim}\, x_n + \underline{\lim}\, y_n = \sup_n \left[\inf_{m \geq n} x_m\right] + \sup_n \left[\inf_{m \geq n} y_m\right]$$

$$= \sup_n \{s_n\} + \sup_n \{t_n\}$$

$$= \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n$$

$$= \lim_{n \to \infty} (s_n + t_n)$$

$$\leq \lim_{n \to \infty} \inf_{m \geq n} (x_m + y_m) \quad [\text{from (1)}]$$

$$= \sup_{n} \left[\inf_{m \geq n} (x_m + y_m) \right]$$

$$= \underline{\lim} (x_n + y_n)$$

$$\underline{\lim}\, x_n + \underline{\lim}\, y_n \leq \underline{\lim}\, (x_n + y_n) \quad \text{— (2)}$$

Now $\overline{\lim}\, x_n = \inf_{n} \left[\sup_{m \geq n} x_m \right]$

$$\overline{\lim}\, y_n = \inf_{n} \left[\sup_{m \geq n} y_m \right]$$

$$p_n = \sup_{m \geq n} x_m \quad \text{and} \quad q_n = \sup_{m \geq n} y_m$$

$$p_n \geq x_m \ \forall\, m \geq n \quad \text{and} \quad q_n \geq y_m \ \forall\, m \geq n$$

$$p_n + q_n \geq x_m + y_m \quad \forall\, m \geq n.$$

$$p_n + q_n \geq \sup_{m \geq n} (x_m + y_m) \quad \text{— ③}$$

$$\overline{\lim}\, x_n + \overline{\lim}\, y_n = \inf_n \left[\sup_{m \geq n} x_m\right] + \inf_n \left[\sup_{m \geq n} y_m\right]$$

$$= \inf_n \{p_n\} + \inf_n \{q_n\}$$

$$= \lim_{n \to \infty} p_n + \lim_{n \to \infty} q_n$$

$$= \lim_{n \to \infty} (p_n + q_n)$$

$$\geq \lim_{n \to \infty} \sup_{m \geq n} (x_m + y_m) \quad [\text{from ③}]$$

$$= \inf_{n} \left[\sup_{m \geq n} (x_m + y_m) \right]$$

$$= \overline{\lim} (x_n + y_n)$$

$$\overline{\lim}\, x_n + \overline{\lim}\, y_n \geq \overline{\lim} (x_n + y_n) \quad \text{--- (4)}$$

By def

$$\underline{\lim} (x_n + y_n) \leq \overline{\lim} (x_n + y_n) \quad \text{--- (5)}$$

$$\underline{\lim}\, x_n + \underline{\lim}\, y_n \leq \underline{\lim} (x_n + y_n) \leq \overline{\lim} (x_n + y_n)$$

$$\leq \overline{\lim}\, x_n + \overline{\lim}\, y_n \quad [\text{from (2), (4) and (5)}]$$

Hence proved.