Continuity of a function

$f_1, f_2, f_3 \ldots\ldots f_k$ be real Valued functions on metric space (X, d) and $f: X \to R^k$ be Vector Valued function defined as

$f(x) = (f_1(x), f_2(x), f_3(x) \ldots\ldots f_k(x)) \ \forall x \in X.$

then f is Continous iff $f_1, f_2, f_3 \ldots\ldots f_k$ is Continous.

Proof

f is Continous. $f: X \to R^k$

for $c \in X$ & $\epsilon > 0$ $\exists \delta > 0$ s.t.

$|f(x) - f(c)| < \epsilon$ when $d(x, c) < \delta$

$$\sqrt{\sum_{i=1}^{k} [f_i(x) - f_i(c)]^2} < \epsilon \quad \text{when } d(x,c) < \delta$$

$$|f_i(x) - f_i(c)| \leq \sqrt{\sum_{i=1}^{k} [f_i(x) - f_i(c)]^2} < \epsilon$$

$$|f_i(x) - f_i(c)| < \epsilon \quad \text{when } d(x,c) < \delta$$
$$\forall i$$

$\Rightarrow$ f_i's are Continous at c.

Hence each f_i is Continous on x.

$\Rightarrow$ $f_1, f_2, f_3 \ldots f_k$ are Continous.

Converse $f_1, f_2, f_3 \ldots\ldots f_k$ are Continous

T.P. f is Continous

$f_i : X \to R^k$ is Continous.

By def.

for $\epsilon > 0$ $\exists$ $\delta > 0$ s.t.

$|f_i(x) - f_i(c)| < \epsilon/k$ when $d(x,c) < \delta$

$\forall i$ — (1)

$\delta = \min\{\delta_1, \delta_2 \ldots \delta_k\}$

$$|f(x) - f(c)|^2 = \sum_{i=1}^{k} [f_i(x) - f_i(c)]^2$$

$$\leq \left\{\sum_{i=1}^{k} f_i(x) - f_i(c)\right]^2$$

$$|f(x) - f(c)| \leq \sum_{i=1}^{k} f_i(x) - f_i(c) < k \cdot \frac{\epsilon}{k} = \epsilon$$

$$|f(x) - f(c)| < \epsilon \quad \text{when } d(x,c) < \delta$$

By def. of Continuity.

$f(x)$ is Continous function.

Hence proved.