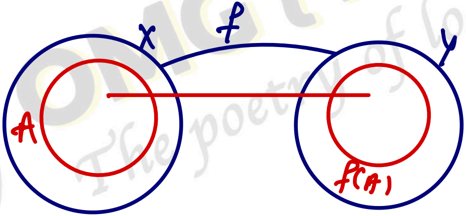
# Continuity of a function

Let $X$ and $Y$ are metric spaces

$f: X \to Y$ is continous iff

$$f(\overline{A}) \subseteq \overline{f(A)} \quad \forall A \subseteq X.$$




Proof

$f: X \to Y$ is continous

$\overline{f(A)}$ is closed set in $Y$

$f: X \to Y$ is continous iff $f^{-1}(C)$ is closed set in $X$ for closed set $C$ in $Y$.

$\Rightarrow$ $f^{-1}(\overline{f(A)})$ is closed in $X$.

$\Rightarrow$ $\overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$ — (1)

Now

$f(A) \subseteq \overline{f(A)}$

$\Rightarrow$ $A \subseteq f^{-1}(\overline{f(A)})$

$\Rightarrow$ $\bar{A} \subseteq \overline{f^{-1}(\overline{f(A)})}$

$\Rightarrow$ $\bar{A} \subseteq f^{-1}(\overline{f(A)})$ [from (1)]

$\Rightarrow$ $f(\bar{A}) \subseteq f\{f^{-1}(\overline{f(A)})\}$.

$$\Rightarrow f(\bar{A}) \subseteq \overline{f(A)} \quad \text{--- (2)}$$

Converse Let $C$ is closed set in $Y$.

$$f^{-1}(C) \subseteq X.$$

$$\Rightarrow f\left(\overline{f^{-1}(C)}\right) \subseteq \overline{f(f^{-1}(C))} \quad \left[\begin{array}{l}\text{from (2)} \\ A = f^{-1}(C)\end{array}\right]$$

$$\Rightarrow f\left(\overline{f^{-1}(C)}\right) \subseteq \bar{C} = C \quad [\because C \text{ is closed}]$$

$$\Rightarrow f\left(\overline{f^{-1}(C)}\right) \subseteq C$$

$$\Rightarrow \overline{f^{-1}(C)} \subseteq f^{-1}(C)$$

Also $f^{-1}(C) \subseteq \overline{f^{-1}(C)}$

$$\overline{f^{-1}(c)} = f^{-1}(c)$$

$f^{-1}(c)$ is closed set in $x$.

$c$ is closed in $Y$

$f^{-1}(c)$ is closed in $X$.

$\Rightarrow$ $f: X \to Y$ is continous

Hence Proved