Continuity of a function

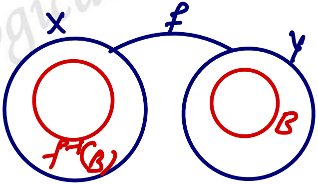
Let X and Y are metric spaces $f: X \to Y$ is Continous function iff

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \quad \forall\, B \subseteq Y.$$

Proof

Let $f: X \to Y$ is Continous

$\overline{B}$ is closed set in Y.



$\Rightarrow f^{-1}(\overline{B})$ is closed in X

[$f: X \to Y$ is Continous iff $f^{-1}(C)$ is closed in X when C is closed in Y]

$$\overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B}) \quad \text{--- (1)}$$

Now $B \subseteq \overline{B}$

$\Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$

$\Rightarrow \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})}$

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ [from ①]

Converse $\quad \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

T.P. f is Continous.

Let C is closed set in Y.

$\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C})$

$\Rightarrow \overline{f^{-1}(c)} \subseteq f^{-1}(c)$ $[\because \bar{c} = c,\ c$ is closed$]$

$\Rightarrow \overline{f^{-1}(c)} = f^{-1}(c)$ $\quad f^{-1}(c) \subseteq \overline{f^{-1}(c)}$

$\Rightarrow f^{-1}(c)$ is a closed set.

$f: X \to Y$ is Continous iff $f^{-1}(c)$ is closed set in X for a closed set c in Y.

$\Rightarrow f$ is Continous function

Hence Proved