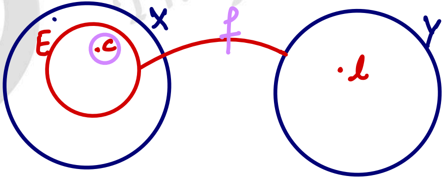
Limit of a function

Let (X, d_1) and (Y, d_2) be two metric spaces.

Let $E \subseteq X$ and c be limit point of E

Let $f : E \to Y$ be a function.

$\lim_{x \to c} f(x) = l$ iff every sequence $\{x_n\}$ in $E - \{c\}$ with $\{x_n\} \to c$, $\{f(x_n)\} \to l$.



Proof $\lim_{x \to c} f(x) = l$

x_n is a sequence in $E - \{c\}$

with $\{x_n\} \to c$.

T.P. $\{f(x_n)\} \to l$.

$\lim_{x \to c} f(x) = l$.

By def. of limit of function

for $\epsilon > 0$ $\exists$ $\delta > 0$

s.t. $d_2[f(x), l] < \epsilon$

when $d_1(x, c) < \delta$ — (1)

$\{x_n\} \to c$

By def. of Convergence of sequence

for $\delta > 0$ $\exists$ $m \in \mathbb{N}$

s.t. $d_1(x_n, c) < \delta$ $\forall$ $n \geq m$.

from ①

$d_2[f(x_n), l] < \epsilon$ $\forall$ $n \geq m$.

By def. of Convergence

$\{f(x_n)\} \to l$.

Converse part

$\forall$ $\{x_n\}$ in $E - \{c\}$

$\{x_n\} \to c$

$\{f(x_n)\} \to l$. — ②

T.P. $\lim_{x \to c} f(x) = l.$

Let $\lim_{x \to c} f(x) \neq l.$

By def. of limit.

for $\epsilon > 0$ $\exists \delta > 0$

s.t. $d_1(x, c) < \delta$

$d_2(f(x), l) \not< \epsilon$

$d_2(f(x), l) \geq \epsilon$

$d_1(x_n, c) < \delta$ and $d_2(f(x_n), l) \geq \epsilon$

$\Rightarrow$ $\{x_n\} \to c$ and $\{f(x_n)\} \not\to l.$

Which is Contradiction of 2

$$\Rightarrow \lim_{x \to c} f(x) = l$$

Hence Proved.