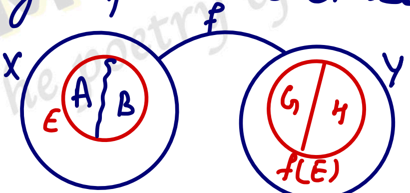
Continuity and Connectedness

Let X and Y be metric spaces. $f: X \to Y$ be Continous function. If E is Connected subset of X then $f(E)$ is Connected subset of Y.

(or)

Continous image of Connected set is Connected.



Proof Let $f(E)$ is disconnected.

$\Rightarrow \exists$ two non-empty, disjoint and separated sets

G & H s.t.

$$f(E) = G \cup H. \quad -①$$

Let $A = f^{-1}(G) \cap E$ $\qquad B = f^{-1}(H) \cap E$.

$$A \cup B = \left(f^{-1}(G) \cap E\right) \cup \left(f^{-1}(H) \cap E\right)$$

$$= \left(f^{-1}(G) \cup f^{-1}(H)\right) \cap E$$

$$= \left(f^{-1}(G \cup H)\right) \cap E$$

$$= \left(f^{-1}(f(E))\right) \cap E$$

$$= E \cap E = E.$$

$A \cup B = E$.

$$\bar{A}\cap B = \;?$$

$$\bar{A}\cap B = \left(\overline{f^{-1}(G)\cap E}\right) \cap \left(f^{-1}(H)\cap E\right)$$

$$\subseteq \left(\overline{f^{-1}(G)} \cap \bar{E}\right) \cap \left(f^{-1}(H)\cap E\right) \qquad \left[\overline{A\cap B}\subseteq \bar{A}\cap\bar{B}\right]$$

$$= \left(\overline{f^{-1}(G)} \cap f^{-1}(H)\right) \cap \left(\bar{E}\cap E\right) \qquad \left[E\subseteq \bar{E}\right]$$

$$= \overline{f^{-1}(G)} \cap f^{-1}(H)\cap E$$

$$\subseteq f^{-1}(\bar{G}) \cap f^{-1}(H)\cap E \qquad \left[\because f \text{ is continous}, \ \overline{f^{-1}(G)}\subseteq f^{-1}(\bar{G})\right]$$

$$= \left(f^{-1}(\bar{G}\cap H)\right)\cap E$$

$$= f^{-1}(\phi)\cap E = \phi \qquad \left\{\bar{G}\cap H=\phi \text{ as } G \text{ and } H \text{ are separated}\right\}$$

$\bar{A} \cap B = \phi$.

Illy $A \cap \bar{B} = \phi$.

$\Rightarrow$ A and B are separated sets.

also $A \cup B = E$.

$\Rightarrow$ E is disconnected.

But E is connected (Given)

Contradiction

So $f(E)$ is connected.

Hence Proved.