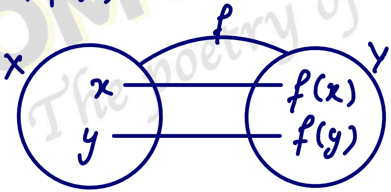
Uniform Continuity

Let X and Y be two metric spaces. Then a function $f: X \to Y$ is said to be uniformly Continous if for every $\epsilon > 0$ $\exists\ \delta > 0$ s.t. $d_Y(f(x), f(y)) < \epsilon$ when $d_X(x,y) < \delta$ $\forall\ x, y \in X$.



Let X and Y be metric spaces and $f: X \to Y$ be continous function. If X is compact then f is uniformly continous

(or)

A continous function defined on a compact set in uniformly continous.

Proof

$f: X \to Y$ is continous

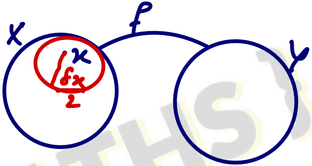
By def.

for $\epsilon > 0$ $\exists$ $\delta_x > 0$ s.t.

$x \in X$ $d_Y(f(x), f(y)) < \epsilon/2$ when $d(x,y) < \delta_x$ — (1)

$x \in X$

$B\left(x, \frac{\delta_x}{2}\right)$ is open cover of X.



X is compact (given)

$\therefore \exists$ a finite subcover

Let $B\left(x_i, \frac{\delta_{x_i}}{2}\right)$ is finite subcover of X

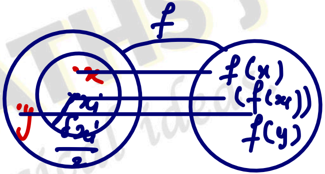
$\{1 \leq i \leq n\}$

$x, y \in X$ $\quad d_X(x, y) < \delta$

$x \in X$ $\quad B\left(x_i, \frac{\delta_{x_i}}{2}\right)$ is open cover of X

$\Rightarrow x \in B\left(x_i, \frac{\delta x_i}{2}\right) \quad 1 \leq i \leq n$

$\Rightarrow d_X(x_i, x) < \frac{\delta x_i}{2} < \delta x_i$



$\Rightarrow d_Y(f(x_i), f(x)) < \frac{\epsilon}{2}$ when $d(x_i, x) < \delta x_i$ $\boxed{\text{from } ①}$

$d_X(y, x_i) \leq d(x_i, x) + d(x, y)$

$< \delta x_i + \delta \leq \frac{\delta x_i}{2} + \frac{\delta x_i}{2} = \delta x_i$

$d_X(y, x_i) < \delta x_i \quad \{1 \leq i \leq n\}$

from ① $\Rightarrow d_Y(f(y), f(x_i)) < \epsilon/2$

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y))$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$d_Y(f(x), f(y)) < \epsilon$. when $d(x,y) < \delta$.

By def. f is uniformly continous