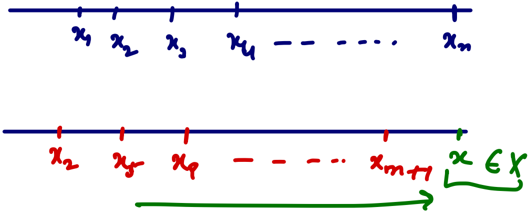
Sequences in metric space

Sub-sequential limit:-

$\{x_n\}$ be a sequence in metric (X, d) then $x \in X$ is called sub-sequential limit of $\{x_n\}$ if $\exists$ a sub sequence $\{x_{n_i}\}$ of $\{x_n\}$ s.t $\{x_{n_i}\} \to x$.



Thm The set of sub-sequential limits of a sequence $\{x_n\}$ in a metric space form a closed subset of X.

Proof: $E = \{x \in X : x \text{ is sub-sequential limit of } \{x_n\}\}$

If E is finite then clearly E is closed.

If E is not finite

To prove E is closed we have to prove that

$$E' \subseteq E$$

Let $x \in E'$ s.t x is limit point of E.

$\Rightarrow$ we find $n_1 \in \mathbb{N}$ s.t.

$$x_{n_1} \neq x$$

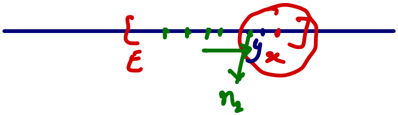
$$d(x_{n_1}, x) < \epsilon$$

Again $x \in E'$

$\therefore \exists$ an open Ball $B\left(x, \frac{\epsilon}{2^2}\right)$ Contains infinitely many points of E.

Let $y \in E$ s.t

$$d(x, y) < \frac{\epsilon}{2^2}$$



$\therefore$ as $y \in E$

y is sub-sequential limit of $\{x_n\}$

$\Rightarrow$ we find a subsequence of x_n converges to y.

$\therefore \exists$ a +ve integer $n_2 > n_1$ s.t

$$d(x_{n_2}, y) < \frac{\epsilon}{2^2}$$

$$d(x_{n_2}, x) \leq d(x_{n_2}, y) + d(y, x)$$

$$< \frac{\epsilon}{2^2} + \frac{\epsilon}{2^2} = \frac{\epsilon}{2}$$

$$d(x_{n_2}, x) < \frac{\epsilon}{2}$$

$\vdots$

By continuing this process we find a sequence of +ve integers s.t $n_1 < n_2 < n_3 - - \cdots < n_i < - - - \cdots$

s.t. $d(x_{n_i}, x) < \frac{\epsilon}{2^{i-1}}$

Now $\frac{\epsilon}{2^{i-1}} \to 0$ as $i \to \infty$

$\therefore$ By def of Convergence of Sequence

$\{x_{n_i}\} \to x \in E'$

$\Rightarrow$ x is subsequential limit

$\Rightarrow$ $x \in E$

$\Rightarrow$ $E' \subseteq E$ $\Rightarrow$ E is a closed set