Limit Inferior and Limit Superior

$\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Then

$\overline{\lim}\, x_n = x$ iff $\forall\, \epsilon > 0$ the following conditions hold.

(a) $x_n > x + \epsilon$ for atmost finitely many terms.

(b) $x_n > x - \epsilon$ for infinitely many terms.

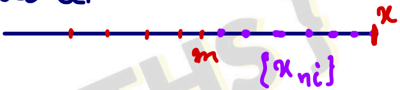
Proof (a) $\overline{\lim}\, x_n = x$.

if possible let $x_n > x + \epsilon$ for infinite many terms.

then we can find a subsequence

$\{x_{n_i}\}$ of $\{x_n\}$ s.t. $x_{n_i} > x + \epsilon \;\forall\, i$

$\Rightarrow$ $\{x_{n_i}\}$ is Bounded Subsequence.



Every Bounded subsequence in $\mathbb{R}^k$ Contains a Convergent subsequence

$\Rightarrow$ $\{x_{n_{i_k}}\}$ is Convergent Subsequence in $\{x_{n_i}\}$

$\Rightarrow$ $\{x_{n_{i_k}}\}$ is Convergent Subsequence of $\{x_n\}$

$$x_{n_{i_k}} > x+\epsilon \quad \forall k$$

Let $\{x_{n_{i_k}}\} \longrightarrow x'$

$$x_{n_{i_k}} > x+\epsilon \ \forall k \Rightarrow x' \geq x+\epsilon$$

$x' > x$

But x' is subsequencial limit of x_n.
which is greater than x
But $\overline{\lim}\, x_n = x$.
which is not possible
Hence $x_n > x + \epsilon$ for atmost finitely many terms

(b) $\overline{\lim}\, x_n = x$.

T.P. $x_n > x - \epsilon$ for infinitely many terms

x is subsequencial limit of x_n

$\Rightarrow$ for $\epsilon > 0$ $|x_n - x| < \epsilon$ for infinitely many terms

$$|x_n - x| < \epsilon$$

$$-\epsilon < x_n - x < \epsilon$$

$$x - \epsilon < x_n < x + \epsilon$$

$\underline{x_n > x - \epsilon}$ for infinite many terms.

<u>Converse</u>:- (a) and (b) satisfied.

<u>T.P</u> $\overline{\lim}\, x_n = x$.

from (a) and (b)

$x - \epsilon < x_n < x + \epsilon$ for infinite many terms.

$$-\epsilon < x_n - x < \epsilon$$

$$|x_n - x| < \epsilon \quad \text{for } \underline{\epsilon > 0}$$

$\Rightarrow$ x is subsequencial limit of x_n

Let x' is also subsequencial limit of x_n

$\boxed{\lim x_n = x}$ we have to show that $x' < x$.

Let $x' > x$

Let $\epsilon = \dfrac{x' - x}{2} > 0$

from (a) $x_n > x + \epsilon$ for atmost finitely many terms.

$$x_n > x + \left(\frac{x' - x}{2}\right) = \frac{2x + x' - x}{2} = \frac{x + x'}{2}$$

$$x_n > \frac{x + x'}{2} \quad \text{[for atmost finitely many terms]} \qquad \text{— (1)}$$

x' is subsequencial limit of x_n

$\Rightarrow$ for $\epsilon > 0$

$$|x_n - x'| < \epsilon \quad \text{for infinite many } n$$

$$-\epsilon < x_n - x' < \epsilon$$

$$x' - \epsilon < x_n < x' + \epsilon$$

$x_n > x' - \epsilon$ for infinite many n

$$x_n > x' - \left(\frac{x' - x}{2}\right)$$

$$= \frac{2x' - x' + x}{2} = \frac{x' + x}{2}$$ for infinite many n.

$x_n > \frac{x' + x}{2}$ for infinite many n.

Which is Contradiction of (i)

$\therefore \quad x' < x$

$\underline{\lim} \, x_n = x.$ Hence Proved.